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The effects of combined horizontal and vertical heterogeneity and anisotropy on the onset of convection in a porous medium

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Abstract

The effects of both horizontal and vertical hydrodynamic and thermal heterogeneity together with anisotropy of both permeability and thermal conductivity, on the onset of convection in a horizontal layer of a saturated porous medium, uniformly heated from below, are studied analytically using linear stability theory for the case of weak heterogeneity. It is found that the effect of such heterogeneity on the critical value of the Rayleigh number *Ra* based on mean properties is of second order if the properties vary in a piecewise constant or linear fashion. The effects of horizontal heterogeneity and vertical heterogeneity are then comparable once the aspect ratio is taken into account, and to a first approximation are independent. For a square enclosure, horizontal heterogeneity is invariably destabilizing, but vertical heterogeneity can be either stabilizing or destabilizing. For an enclosure whose aspect ratio is optimized to give the minimum value of the critical Rayleigh number, both horizontal and vertical heterogeneity are destabilizing, by an amount dependent on the ratio of the conductivity and permeability anisotropy measures.
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1. Introduction

The problem of the onset of convection in a horizontal layer of fluid heated uniformly from below is commonly called the Rayleigh–Bénard problem in the case of a fluid clear of solid material and the Horton–Rogers–Lapwood (HRL) problem for the case of a fluid-saturated porous medium. The latter is treated in some detail in the book by Nield and Bejan [1].

In recent discussions about the effect of heterogeneity (of either permeability or thermal conductivity or both) on convection in a porous medium it has been noted that in the case of strong heterogeneity there can be dramatic effects (Simmons et al. [2], Prasad and Simmons [3], Nield and Simmons [4]). Even in the case of weak heterogeneity it is of interest to investigate the combined effects of vertical heterogeneity (property variation in the vertical direction, including horizontal layering as a special case) and horizontal heterogeneity. This is the subject of the analysis of Nield and Kuznetsov [5]. The survey of the effects of heterogeneity in the book by Nield and Bejan [1] indicates this topic had not been considered previously. In their analytical study Nield and Kuznetsov [5] found that the effect of such heterogeneity on the critical value of the Rayleigh number *Ra* based on mean properties is of second order if the properties vary in a piecewise constant or linear fashion. The effects of horizontal heterogeneity and vertical heterogeneity are then comparable and to a first approximation are independent. For the case of conducting impermeable top and bottom boundaries and a square box, the effects of permeability heterogeneity and conductivity permeability each cause a reduction in the critical value of *Ra*, while for the case of a tall box there can be either a reduction or an increase. It was found by Nield and Kuznetsov [6] that in the case of a shallow box with constant-flux top and bottom boundaries there can be either a reduction or increase in the critical value of the Rayleigh number.

In the present paper the analysis of Nield and Kuznetsov [5] is extended to the case where both the permeability and the thermal conductivity are anisotropic. (The same methodology is employed.) Previous work on the effect of anisotropy is sur-

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Nomenclature

A	height-to-width aspect ratio, H/L
g	gravity
H	height of the enclosure
$k_H^*(x^*, y^*)$	overall (effective) thermal conductivity in the horizontal direction
$k_V^*(x^*, y^*)$	overall (effective) thermal conductivity in the vertical direction
k_{0H}	arithmetic mean value of $k_H^*(x^*, y^*)$
k_{0V}	arithmetic mean value of $k_V^*(x^*, y^*)$
$K_H^*(x^*, y^*)$	permeability in the horizontal direction
$K_V^*(x^*, y^*)$	permeability in the vertical direction
K_{0H}	harmonic mean value of $K_H^*(x^*, y^*)$
K_{0V}	harmonic mean value of $K_V^*(x^*, y^*)$
$K^*(x^*, y^*)$	permeability
L	enclosure width
P	dimensionless pressure, $\frac{(\rho c)_m K_{0V}}{\mu k_{0V}} P^*$
P^*	pressure
Ra	Rayleigh number, $\frac{(\rho c)_f \rho_0 g \beta K_{0V} H(T_1 - T_0)}{\mu k_{0V}}$
Ra_0	Rayleigh number for the homogeneous case, defined by Eq. (43)
S	parameter defined by Eq. (53)
t^*	time
t	dimensionless time, $\frac{k_{0V}}{(\rho c)_m H^2} t^*$
T^*	temperature
T_1	temperature at the lower boundary
T_0	temperature at the upper boundary
u	dimensionless horizontal velocity, $\frac{(\rho c)_m L}{k_{0V}} u^*$
u^*	dimensional horizontal velocity
$\mathbf{u}^*$	vector of Darcy velocity, (u^*, v^*)
v	dimensionless vertical velocity, $\frac{(\rho c)_m H}{k_{0V}} v^*$
v^*	dimensional vertical velocity
x	dimensionless horizontal coordinate, x^*/L
x^*	horizontal coordinate
y	dimensionless upward vertical coordinate, y^*/H
y^*	upward vertical coordinate

Greek symbols

β	volumetric thermal expansion coefficient of the fluid
δ_{Hx}	variation in the x-direction of horizontal permeability (Eq. (46))
ε_{Hx}	variation in the x-direction of horizontal conductivity (Eq. (46))
η	anisotropy conductivity ratio, $\frac{k_{H0}}{k_{V0}}$
θ	dimensionless temperature, $\frac{T^* - T_0}{T_1 - T_0}$
ξ	anisotropy permeability ratio, $\frac{K_{H0}}{K_{V0}}$
ρ_0	fluid density at temperature T_0
$(\rho c)_f$	heat capacity of the fluid
$(\rho c)_m$	heat capacity of the overall porous medium
σ	heat capacity ratio, $\frac{(\rho c)_m}{(\rho c)_f}$
ψ	streamfunction defined by Eq. (13)

Superscripts

$*$	dimensional variable

veyed in Section 6.12 of [1] while previous work on the effect of heterogeneity is surveyed in Section 6.13 of that book. Since over 25 papers on each topic have been published they are not individually cited here.

Our main interest is in fundamental rather than specific problems, but we note that our results should be readily applicable to geophysical situations, in which vertical/horizontal anisotropy and heterogeneity characteristically occur together.

2. Analysis

Single-phase flow in a saturated porous medium is considered. Asterisks are used to denote dimensional variables. We consider a rectangular box, $0 \leqslant x^* \leqslant L$, $0 \leqslant y^* \leqslant H$, where the y^* axis is in the upward vertical direction. The side walls are taken as insulated, and uniform temperatures T_0 and T_1 are imposed at the upper and lower boundaries, respectively.

Within this box the permeability is assumed to have horizontal component $K_H^*(x^*, y^*)$ and vertical component $K_V^*(x^*, y^*)$ and likewise the horizontal and vertical components of the overall (effective) thermal conductivity are given by $k_H^*(x^*, y^*)$ and $k_V^*(x^*, y^*)$. The Darcy velocity is denoted by $\mathbf{u}^* = (u^*, v^*)$. The Oberbeck–Boussinesq approximation is invoked. The equations representing the conservation of mass, thermal energy, and Darcy's law take the form

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \tag{1}$$

$$(\rho c)_m \frac{\partial T^*}{\partial t^*} + (\rho c)_f \left[u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} \right] = k_H^*(x^*, y^*) \frac{\partial^2 T^*}{\partial x^{*2}} + k_V^*(x^*, y^*) \frac{\partial^2 T^*}{\partial y^{*2}} \tag{2}$$

$$u^* = -\frac{K_H^*(x^*, y^*)}{\mu} \frac{\partial P^*}{\partial x^*} \tag{3a}$$

$$v^* = \frac{K_V^*(x^*, y^*)}{\mu} \left[-\frac{\partial P^*}{\partial y^*} - \rho_0 \beta g (T^* - T_0) \right] \tag{3b}$$

Here $(\rho c)_m$ and $(\rho c)_f$ are the heat capacities of the overall porous medium and the fluid, respectively, μ is the fluid viscosity, ρ_0 is the fluid density at temperature T_0, and β is the volumetric expansion coefficient, while T^* is the temperature in the porous medium. (Local thermal equilibrium between solid and fluid phases is assumed.)

In order to simplify the following analysis, on the right-hand side of Eq. (2) the terms involving the partial derivatives of k_H^* and k_V^* with respect to the spatial coordinates have been dropped. In accordance with the assumption of weak heterogeneity, it is assumed that the variation of each of these quantities over the enclosure is small compared with the mean value of the quantity. Thus, for example, $(L/k_H^*)\partial k_H/\partial x^*$ is assumed to be small compared with unity. It can be shown that this approximation has no effect on the results presented in this paper provided that each quantity is a linear function of the spatial variables considered separately. A similar assumption about the variation of the permeability is made below.

We introduce dimensionless variables by defining

$$x = \frac{x^*}{L}, \qquad y = \frac{y^*}{H}, \qquad u = \frac{(\rho c)_m L}{k_{0V}} u^*$$
$$v = \frac{(\rho c)_m H}{k_{0V}} v^*, \qquad t = \frac{k_{0V}}{(\rho c)_m H^2} t^*$$
$$\theta = \frac{T^* - T_0}{T_1 - T_0}, \qquad P = \frac{(\rho c)_m K_{0V}}{\mu k_{0V}} P^* \tag{4}$$

where k_{0V} is the arithmetic mean value of $k_V^*(x^*, y^*)$ and K_{0V} is the harmonic mean value of $K_V^*(x^*, y^*)$. The quantities k_{0H} and K_{0H} are defined in a similar fashion. The choice of harmonic mean for the permeabilities and arithmetic mean for the conductivities is made because of the way in which those quantities enter the governing differential equations (compare Eq. (14) with Eq. (15)).

We also define a Rayleigh number Ra by

$$Ra = \frac{(\rho c)_f \rho_0 g \beta K_{0V} H (T_1 - T_0)}{\mu k_{0V}} \tag{5}$$

the heat capacity ratio by

$$\sigma = \frac{(\rho c)_m}{(\rho c)_f} \tag{6}$$

the height-to-width aspect ratio by

$$A = H/L \tag{7}$$

and the anisotropy permeability and conductivity ratios ξ and η by

$$\xi = \frac{K_{H0}}{K_{V0}} \tag{8a}$$

$$\eta = \frac{k_{H0}}{k_{V0}} \tag{8b}$$

The governing equations then take the form

$$A^2 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

$$\frac{\partial \theta}{\partial t} + \frac{1}{\sigma}\left[A^2 u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y}\right] = \eta k_H(x, y) A^2 \frac{\partial^2 \theta}{\partial x^2} + k_V(x, y) \frac{\partial^2 \theta}{\partial y^2} \tag{10}$$

$$u = -\xi K_H(x, y) \frac{\partial P}{\partial x}$$
$$v = K_V(x, y)\left[-\frac{\partial P}{\partial y} + \sigma Ra \theta\right] \tag{11}$$

where

$$k_H = k_H^*(x^*, y^*)/k_{H0} \tag{12a}$$
$$k_V = k_V^*(x^*, y^*)/k_{V0} \tag{12b}$$
$$K_H = K_H^*(x^*, y^*)/K_{H0} \tag{12c}$$
$$K_V = K_V^*(x^*, y^*)/K_{V0} \tag{12d}$$

We introduce a streamfunction ψ so that

$$u = \frac{\sigma Ra}{A^2} \frac{\partial \psi}{\partial y} \tag{13a}$$

$$v = -\sigma Ra \frac{\partial \psi}{\partial x} \tag{13b}$$

Then Eq. (9) is satisfied identically. We also eliminate P by combining Eq. (11) with Eqs. (13a), (13b). In doing this we assume that, in accordance with the assumption of weak heterogeneity, that the maximum variation of K_H over the domain is small compared with the mean value of K_H, so we can approximate $\partial(u/K_H)/\partial x$ by $(1/K_H)\partial u/\partial x$, etc. The result is

$$\frac{A^2}{K_V(x, y)} \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{\xi K_H(x, y)} \frac{\partial^2 \psi}{\partial y^2} = -A^2 \frac{\partial \theta}{\partial x} \tag{14}$$

$$\frac{\partial \theta}{\partial t} + Ra\left[\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y}\right] = \eta k_H(x, y) A^2 \frac{\partial^2 \theta}{\partial x^2} + k_V(x, y) \frac{\partial^2 \theta}{\partial y^2} \tag{15}$$

The standard linear stability analysis is now employed. The basic (conduction) solution is perturbed and the equations are linearized.

The conduction solution is given by

$$\psi = 0 \tag{16a}$$
$$\theta = 1 - y \tag{16b}$$

The perturbed solution is given by

$$\psi = \varepsilon \psi' \tag{17a}$$
$$\theta = 1 - y + \varepsilon \theta' \tag{17b}$$

To first order in the small constant ε, we get

$$\frac{A^2}{K_V(x, y)} \frac{\partial^2 \psi'}{\partial x^2} + \frac{1}{\xi K_H(x, y)} \frac{\partial^2 \psi'}{\partial y^2} + A^2 \frac{\partial \theta'}{\partial x} = 0 \tag{18}$$

$$\frac{\partial \theta'}{\partial t} + Ra \frac{\partial \psi'}{\partial x} - \eta k_H(x, y) A^2 \frac{\partial^2 \theta'}{\partial x^2} - k_V(x, y) \frac{\partial^2 \theta'}{\partial y^2} = 0 \tag{19}$$

For the onset of convection we can invoke the "principal of exchange of stabilities" and hence take the time derivative in Eq. (19) to be zero.

The boundary conditions are

$$\psi' = 0 \quad \text{and} \quad \theta' = 0 \quad \text{on } y = 0 \tag{20a,b}$$
$$\psi' = 0 \quad \text{and} \quad \theta' = 0 \quad \text{on } y = 1 \tag{21a,b}$$
$$\psi' = 0 \quad \text{and} \quad \partial\theta'/\partial x = 0 \quad \text{on } x = 0 \tag{22a,b}$$
$$\psi' = 0 \quad \text{and} \quad \partial\theta'/\partial x = 0 \quad \text{on } x = 1 \tag{23a,b}$$

This set of boundary conditions is satisfied by

$$\psi'_{mn} = \sin m\pi x \sin n\pi y, \quad m, n = 1, 2, 3, \ldots \tag{24}$$
$$\theta'_{pq} = \cos p\pi x \sin q\pi y, \quad p, q = 1, 2, 3, \ldots \tag{25}$$

We can take this set of functions (that are exact eigenfunctions for the homogeneous case) as trial functions for an approximate solution of the heterogeneous case. For example, working at second order, we can try

$$\psi' = A_{11}\psi'_{11} + A_{12}\psi'_{12} + A_{21}\psi'_{21} + A_{22}\psi'_{22} \tag{26}$$

$$\theta' = B_{11}\theta'_{11} + B_{12}\theta'_{12} + B_{21}\theta'_{21} + B_{22}\theta'_{22} \tag{27}$$

In the Galerkin method, the expressions (26) and (27) are substituted into the left-hand side of Eq. (18) and the resulting residual is made orthogonal to the separate trial functions ψ'_{11}, ψ'_{12}, ψ'_{21}, ψ'_{22} in turn. Likewise the residual on the substitution of the expressions (26) and (27) into Eq. (19) is made orthogonal to θ'_{11}, θ'_{12}, θ'_{21}, θ'_{22} in turn.

We use the notation

$$\langle f(x,y)\rangle = \int_0^1\int_0^1 f(x,y)\,dx\,dy \tag{28}$$

and define

$$I_{mnpq} = 4\langle [K_H(x,y)]^{-1}\sin m\pi x\sin n\pi y\sin p\pi x\sin q\pi y\rangle \tag{29}$$

$$J_{mnpq} = 4\langle [K_V(x,y)]^{-1}\sin m\pi x\sin n\pi y\sin p\pi x\sin q\pi y\rangle \tag{30}$$

$$K_{mnpq} = 4\langle k_H(x,y)\cos m\pi x\sin n\pi y\cos p\pi x\sin q\pi y\rangle \tag{31}$$

$$L_{mnpq} = 4\langle k_V(x,y)\cos m\pi x\sin n\pi ys\cos p\pi x\sin q\pi y\rangle \tag{32}$$

(Note that the symbol K on the left-hand side of Eq. (31) is not related to permeability.) We note that $\langle k_H(x,y)\rangle = 1$, $\langle k_V(x,y)\rangle = 1$, $\langle [K_H(x,y)]^{-1}\rangle = 1$ and $\langle [K_V(x,y)]^{-1}\rangle = 1$.

Also,

$$4\langle \sin m\pi x\sin n\pi y\sin p\pi x\sin q\pi y\rangle = \begin{cases} 1 & \text{if } m = p \text{ and } n = q \\ 0 & \text{otherwise}\end{cases} \tag{33}$$

$$4\langle \cos m\pi x\sin n\pi y\cos p\pi x\sin q\pi y\rangle = \begin{cases} 1 & \text{if } m = p \text{ and } n = q \\ 0 & \text{otherwise}\end{cases} \tag{34}$$

The output of the Galerkin procedure is a set of 8 homogeneous linear equations in the 8 unknown constants A_{11}, A_{12}, A_{21}, A_{22}, B_{11}, B_{12}, B_{21}, B_{22}. Eliminating these constants we get

$$\det \mathbf{M} = 0 \tag{35}$$

where the matrix $\mathbf{M}$ takes the form

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix} \tag{36}$$

where

$$\mathbf{M}_{11} = \begin{bmatrix} \pi^2(A^2J_{1111}+\xi^{-1}I_{1111}) & \pi^2(A^2J_{1211}+4\xi^{-1}I_{1211}) & \pi^2(4A^2J_{2111}+\xi^{-1}I_{2111}) & \pi^2(4A^2J_{2211}+4\xi^{-1}I_{2211}) \\ \pi^2(A^2J_{1112}+\xi^{-1}I_{1112}) & \pi^2(A^2J_{1212}+4\xi^{-1}I_{1212}) & \pi^2(4A^2J_{2112}+\xi^{-1}I_{2112}) & \pi^2(4A^2J_{2212}+4\xi^{-1}I_{2212}) \\ \pi^2(A^2J_{1121}+\xi^{-1}I_{1121}) & \pi^2(A^2J_{1221}+4\xi^{-1}I_{1221}) & \pi^2(4A^2J_{2121}+\xi^{-1}I_{2121}) & \pi^2(4A^2J_{2221}+4\xi^{-1}I_{2221}) \\ \pi^2(A^2J_{1122}+\xi^{-1}I_{1122}) & \pi^2(A^2J_{1222}+4\xi^{-1}I_{1222}) & \pi^2(4A^2J_{2122}+\xi^{-1}I_{2122}) & \pi^2(4A^2J_{2222}+4\xi^{-1}I_{2222}) \end{bmatrix} \tag{37}$$

$$\mathbf{M}_{12} = \begin{bmatrix} \pi A^2 & 0 & 0 & 0 \\ 0 & \pi A^2 & 0 & 0 \\ 0 & 0 & 2\pi A^2 & 0 \\ 0 & 0 & 0 & 2\pi A^2 \end{bmatrix} \tag{38}$$

$$\mathbf{M}_{21} = \begin{bmatrix} \pi Ra & 0 & 0 & 0 \\ 0 & \pi Ra & 0 & 0 \\ 0 & 0 & 2\pi Ra & 0 \\ 0 & 0 & 0 & 2\pi Ra \end{bmatrix} \tag{39}$$

$$\mathbf{M}_{22} = \begin{bmatrix} \pi^2(\eta A^2K_{1111}+L_{1111}) & \pi^2(\eta A^2K_{1211}+4L_{1211}) & \pi^2(4\eta A^2K_{2111}+L_{2111}) & \pi^2(4\eta A^2K_{2211}+4L_{2211}) \\ \pi^2(\eta A^2K_{1112}+L_{1112}) & \pi^2(\eta A^2K_{1212}+4L_{1212}) & \pi^2(4\eta A^2L_{2112}+L_{2112}) & \pi^2(4\eta A^2K_{2212}+4L_{2212}) \\ \pi^2(\eta A^2K_{1121}+L_{1121}) & \pi^2(\eta A^2K_{1221}+4L_{1221}) & \pi^2(4\eta A^2K_{2121}+L_{2121}) & \pi^2(4\eta A^2K_{2221}+4L_{2221}) \\ \pi^2(\eta A^2K_{1122}+L_{1122}) & \pi^2(\eta A^2K_{1222}+4L_{1222}) & \pi^2(4\eta A^2K_{2122}+L_{2122}) & \pi^2(4\eta A^2K_{2222}+4L_{2222}) \end{bmatrix} \tag{40}$$

In the case of arbitrary functions of x and y, the integrals in Eqs. (37) and (40) can be obtained by quadrature. The eigenvalue equation, Eq. (35) can then be solved to give a value of Ra that is a good approximation to the Rayleigh number.

Based on our experience with the Rayleigh–Bénard problem, we expect that the Galerkin approximation will lead to an overestimate of the Rayleigh number by not more than 3%. For a given order of approximation, the amount of error should be fairly uniform as the heterogeneity parameters vary. Hence we expect that the method used will give a quite accurate value of the relative deviation of Ra from its homogeneous case value (the quantity S defined by Eq. (53) below).

3. Results and discussion

3.1. First order results

For example, the order-one Galerkin method (using a single trial function for each of ψ and θ) yields the eigenvalue equation

$$\det\begin{bmatrix} \pi^2(A^2J_{1111}+\xi^{-1}I_{1111}) & A^2\pi \\ \pi Ra & \pi^2(\eta A^2K_{1111}+L_{1111})\end{bmatrix} = 0 \tag{41}$$

which gives

$$Ra = \pi^2\big(A^2J_{1111}+\xi^{-1}I_{1111}\big)\big(\eta A^2K_{1111}+L_{1111}\big)/A^2 \tag{42}$$

For the homogeneous case, $I_{1111} = J_{1111} = K_{1111} = L_{1111} = 1$, and so $Ra = Ra_0$ where

$$Ra_0 = \pi^2\big(A^2+\xi^{-1}\big)\big(1+\eta A^2\big)/A^2 \tag{43}$$

in accord with Eq. (6.131) of Nield and Bejan [1].

As A varies, Ra_0 is a minimum when $A = (\xi\eta)^{-1/4}$, and the minimum value is

$$Ra_{0\,\min} = \pi^2\big[1+(\eta/\xi)^{1/2}\big]^2 \tag{44}$$

In the subsequent discussion Ra_0 is used as the base line for comparison.

3.2. Second order results

In order to examine the interaction of permeability heterogeneity and conductivity heterogeneity we return to Eqs. (35)–(40) and apply these to the quartered square with piecewise-constant properties. We consider the case

$$
\begin{aligned}
&1/K_H(x,y) = 1 - \delta_{Hx} - \delta_{Hy}\\
&k_H(x,y) = 1 - \varepsilon_{Hx} - \varepsilon_{Hy}\\
&1/K_V(x,y) = 1 - \delta_{Vx} - \delta_{Vy}\\
&k_V(x,y) = 1 - \varepsilon_{Vx} - \varepsilon_{Vy}\\
&\quad \text{for } 0 \leqslant x < 1/2,\ 0 \leqslant y < 1/2\\
&1/K_H(x,y) = 1 + \delta_{Hx} - \delta_{Hy}\\
&k_H(x,y) = 1 + \varepsilon_{Hx} - \varepsilon_{Hy}\\
&1/K_V(x,y) = 1 + \delta_{Vx} - \delta_{Vy}\\
&k_V(x,y) = 1 + \varepsilon_{Vx} - \varepsilon_{Vy}\\
&\quad \text{for } 1/2 < x \leqslant 1,\ 0 \leqslant y < 1/2\\
&1/K_H(x,y) = 1 - \delta_{Hx} + \delta_{Hy}\\
&k_H(x,y) = 1 - \varepsilon_{Hx} + \varepsilon_{Hy}\\
&1/K_V(x,y) = 1 - \delta_{Vx} + \delta_{Vy}\\
&k_V(x,y) = 1 - \varepsilon_{Vx} + \varepsilon_{Vy}\\
&\quad \text{for } 0 \leqslant x < 1/2,\ 1/2 < y \leqslant 1\\
&1/K_H(x,y) = 1 + \delta_{Hx} + \delta_{Hy}\\
&k_H(x,y) = 1 + \varepsilon_{Hx} + \varepsilon_{Hy}\\
&1/K_V(x,y) = 1 + \delta_{Vx} + \delta_{Vy}\\
&k_V(x,y) = 1 + \varepsilon_{Vx} + \varepsilon_{Vy}\\
&\quad \text{for } 1/2 < x \leqslant 1,\ 1/2 < y \leqslant 1\\
&1/K_H(x,1/2) = 1 - \delta_{Hx}\\
&k_H(x,1/2) = 1 - \varepsilon_{Hx}\\
&1/K_V(x,1/2) = 1 - \delta_{Vx}\\
&k_V(x,1/2) = 1 - \varepsilon_{Vx}\\
&\quad \text{for } 0 \leqslant x < 1/2\\
&1/K_H(x,1/2) = 1 + \delta_{Hx}\\
&k_H(x,1/2) = 1 + \varepsilon_{Hx}\\
&1/K_V(x,1/2) = 1 + \delta_{Vx}\\
&k_V(x,1/2) = 1 + \varepsilon_{Vx}\\
&\quad \text{for } 1/2 < x \leqslant 1\\
&1/K_H(1/2,y) = 1 - \delta_{Hy}\\
&k_H(1/2,y) = 1 - \varepsilon_{Hy}\\
&1/K_V(1/2,y) = 1 - \delta_{Vy}\\
&k_V(1/2,y) = 1 - \varepsilon_{Vy} \quad \text{for } 0 \leqslant y < 1/2\\
&1/K_H(1/2,y) = 1 + \delta_{Hy}\\
&k_H(1/2,y) = 1 + \varepsilon_{Hy}\\
&1/K_V(1/2,y) = 1 + \delta_{Vy}\\
&k_V(1/2,y) = 1 + \varepsilon_{Vy} \quad \text{for } 1/2 < y \leqslant 1\\
&1/K_H(1/2,1/2) = 1, \qquad k_H(1/2,1/2) = 1\\
&1/K_V(1/2,1/2) = 1, \qquad k_V(1/2,1/2) = 1
\end{aligned}
\tag{45}
$$

This case approximates a general case in which each slowly varying quantity is approximated by a piecewise-constant distribution (one in which the domain is divided into subdomains in each of which the variable takes a constant value). The mean value of the quantity is approximated by its value at centre of the main square:

$$\bar{f} = f(0.5, 0.5)$$

In each quarter, the function is approximated by its value at the centre of that quarter, and a truncated Taylor series expansion is used to approximate this factor. For example, in the region $1/2 < x < 1$, $1/2 < y < 1$, $f(x,y)$ is approximated by $f(0.75, 0.75)$ and then by $f(0.5,0.5) + 0.25 f_x(0.5,0.5) + 0.25 f_y(0.5,0.5)$.

Thus, for example,

$$
\begin{aligned}
&\delta_{Hx} = \frac{1}{4}\left[\frac{1}{(1/K_H)}\frac{\partial(1/K_H)}{\partial x}\right]_{(1/2,1/2)}\\
&\delta_{Hy} = \frac{1}{4}\left[\frac{1}{(1/K_H)}\frac{\partial(1/K_H)}{\partial y}\right]_{(1/2,1/2)}\\
&\varepsilon_{Hx} = \frac{1}{4}\left[\frac{1}{k_H}\frac{\partial k_H}{\partial x}\right]_{(1/2,1/2)}\\
&\varepsilon_{Hy} = \frac{1}{4}\left[\frac{1}{k_H}\frac{\partial k_H}{\partial y}\right]_{(1/2,1/2)}
\end{aligned}
\tag{46}
$$

In terms of the shorthand notation

$$
\begin{aligned}
&\Delta_{Hx} = (8/3\pi)\delta_{Hx}, \qquad \Delta_{Hy} = (8/3\pi)\delta_{Hy}\\
&\mathrm{E}_{Hx} = (4/3\pi)\varepsilon_{Hx}, \qquad \mathrm{E}_{Hy} = (8/3\pi)\varepsilon_{Hy}\\
&\Delta_{Vx} = (8/3\pi)\delta_{Vx}, \qquad \Delta_{Vy} = (8/3\pi)\delta_{Vy}\\
&\mathrm{E}_{Vx} = (4/3\pi)\varepsilon_{Vx}, \qquad \mathrm{E}_{Vy} = (8/3\pi)\varepsilon_{Vy}
\end{aligned}
\tag{47}
$$

one has

$$
\begin{aligned}
&I_{1111} = I_{1212} = I_{2121} = I_{2222} = 1\\
&I_{1211} = I_{1112} = I_{2221} = I_{2122} = -\Delta_{Hy}\\
&I_{2111} = I_{2212} = I_{1121} = I_{1222} = -\Delta_{Hx}\\
&I_{2211} = I_{2112} = I_{1221} = I_{1122} = 0\\
&J_{1111} = J_{1212} = J_{2121} = J_{2222} = 1\\
&J_{1211} = J_{1112} = J_{2221} = J_{2122} = -\Delta_{Vy}\\
&J_{2111} = J_{2212} = J_{1121} = J_{1222} = -\Delta_{Vx}\\
&J_{2211} = J_{2112} = J_{1221} = J_{1122} = 0\\
&K_{1111} = K_{1212} = K_{2121} = K_{2222} = 1\\
&K_{1211} = K_{1112} = K_{2221} = K_{2122} = -\mathrm{E}_{Hy}\\
&K_{2111} = K_{2212} = K_{1121} = K_{1222} = -\mathrm{E}_{Hx}\\
&L_{1111} = L_{1212} = L_{2121} = L_{2222} = 1\\
&L_{1211} = L_{1112} = L_{2221} = L_{2122} = -\mathrm{E}_{Vy}\\
&L_{2111} = L_{2212} = L_{1121} = L_{1222} = -\mathrm{E}_{Vx}\\
&L_{2211} = L_{2112} = L_{1221} = L_{1122} = 0
\end{aligned}
\tag{48}
$$

Table 1
Values of the Rayleigh number coefficients, defined by Eq. (54), for $A = 1$ and various parameter values (ξ, η)

ξ, η	0.1, 0.1	1, 0.1	10, 0.1	0.1, 1	1, 1	10, 1	0.1, 10	1, 10	10, 10
Ra_0	119.42	21.71	11.94	217.13	39.48	21.71	1194.22	217.13	119.42
C_{11x}	0.318	0.280	−0.010	1.819	−0.200	−0.003	−2.984	−0.126	−0.002
C_{22x}	0.013	1.121	−4.076	0.073	−0.801	−1.120	−0.119	−0.505	−0.897
C_{33x}	0.004	0.018	−0.040	0.280	−0.200	−0.126	−1.019	−0.280	−0.224
C_{44x}	0.318	1.819	−2.984	0.280	−0.200	−0.126	−0.010	−0.003	−0.002
C_{12x}	0.159	1.401	−0.509	0.910	−1.001	−0.140	−1.492	−0.631	−0.112
C_{34x}	0.080	0.455	−0.746	0.700	−0.500	−0.316	−0.254	−0.070	−0.056
C_{13x}	0.125	0.200	−0.040	2.002	−0.400	−0.031	−4.002	−0.308	−0.030
C_{24x}	0.250	4.003	−8.006	0.400	−0.801	−0.616	−0.080	−0.062	−0.060
C_{14x}	1.001	1.601	−0.320	1.601	−0.320	−0.246	−0.320	−0.025	−0.002
C_{23x}	0.020	0.320	−0.640	0.320	−0.640	−0.492	−0.640	−0.492	−0.482
C_{11y}	−0.689	−0.323	−0.024	−0.716	−0.343	−0.027	−0.810	−0.420	−0.049
C_{22y}	−0.002	−0.081	−0.593	−0.002	−0.086	−0.682	−0.002	−0.105	−1.223
C_{33y}	−0.001	−0.001	−0.001	−0.020	−0.022	−0.026	−0.148	−0.171	−0.306
C_{44y}	−0.689	−0.716	−0.810	−0.323	−0.343	−0.420	−0.024	−0.027	−0.049
C_{12y}	−0.086	−0.404	−0.297	−0.089	−0.429	−0.341	−0.101	−0.525	−0.611
C_{34y}	−0.043	−0.044	−0.050	−0.202	−0.214	−0.262	−0.148	−0.170	−0.306
C_{13y}	−0.012	−0.010	−0.004	−0.098	−0.086	−0.038	−0.398	−0.376	−0.240
C_{24y}	−0.023	−0.197	−0.795	−0.020	−0.172	−0.075	−0.008	−0.075	−0.480
C_{14y}	−0.369	−0.315	−0.127	−0.315	−0.274	−0.120	−0.127	−0.120	−0.077
C_{23y}	−0.001	−0.004	−0.016	−0.004	−0.343	−0.150	−0.016	−0.150	−0.960

Now one has

$$\mathbf{M}_{11} = \begin{bmatrix} \pi^2(\xi^{-1}+A^2) & -\pi^2(4\xi^{-1}\Delta_{Hy}+A^2\Delta_{Vy}) & -\pi^2(\xi^{-1}\Delta_{Hx}+4A^2\Delta_{Vx}) & 0 \\ -\pi^2(\xi^{-1}\Delta_{Hy}+A^2\Delta_{Vy}) & \pi^2(4\xi^{-1}+A^2) & 0 & -\pi^2(4\xi^{-1}\Delta_{Hx}+4A^2\Delta_{Vx}) \\ -\pi^2(\xi^{-1}\Delta_{Hx}+A^2\Delta_{Vx}) & 0 & \pi^2(\xi^{-1}+4A^2) & -\pi^2(4\xi^{-1}\Delta_{Hy}+4A^2\Delta_{Vy}) \\ 0 & -\pi^2(4\xi^{-1}\Delta_{Hx}+A^2\Delta_{Vx}) & -\pi^2(\xi^{-1}\Delta_{Hy}+4A^2\Delta_{Vy}) & \pi^2(4\xi^{-1}+4A^2) \end{bmatrix} \tag{49}$$

$$\mathbf{M}_{12} = \begin{bmatrix} \pi A^2 & 0 & 0 & 0 \\ 0 & \pi A^2 & 0 & 0 \\ 0 & 0 & 2\pi A^2 & 0 \\ 0 & 0 & 0 & 2\pi A^2 \end{bmatrix} \tag{50}$$

$$\mathbf{M}_{21} = \begin{bmatrix} \pi Ra & 0 & 0 & 0 \\ 0 & \pi Ra & 0 & 0 \\ 0 & 0 & 2\pi Ra & 0 \\ 0 & 0 & 0 & 2\pi Ra \end{bmatrix} \tag{51}$$

$$\mathbf{M}_{22} = \begin{bmatrix} \pi^2(\eta A^2+1) & -\pi^2(\eta A^2\mathrm{E}_{Hy}+4\mathrm{E}_{Vy}) & -\pi^2(4\eta A^2\mathrm{E}_{Hx}+\mathrm{E}_{Vx}) & 0 \\ -\pi^2(\eta A^2\mathrm{E}_{Hy}+\mathrm{E}_{Vy}) & \pi^2(\eta A^2+4) & 0 & -\pi^2(4\eta A^2\mathrm{E}_{Hx}+4\mathrm{E}_{Vx}) \\ -\pi^2(\eta A^2\mathrm{E}_{Hx}+\mathrm{E}_{Vx}) & 0 & \pi^2(4\eta A^2+1) & -\pi^2(4\eta A^2\mathrm{E}_{Hy}+4\mathrm{E}_{Vy}) \\ 0 & -\pi^2(\eta A^2\mathrm{E}_{Hx}+4\mathrm{E}_{Vx}) & -\pi^2(4\eta A^2\mathrm{E}_{Hy}+\mathrm{E}_{Vy}) & \pi^2(4\eta A^2+4) \end{bmatrix} \tag{52}$$

The eigenvalue equation expands to give a quartic equation in Ra, and the smallest root is sought. For the homogeneous case this is the value Ra_0 given by Eq. (43).

Consider the case where $\Delta_{Hx}, \ldots, \mathrm{E}_{Vy}$ are all small compared with unity.

One can now set

$$Ra = Ra_0(1+S) \tag{53}$$

where S is small compared with unity. Substituting, linearizing, and solving for S one obtains an expression for S that, when substituted back into Eq. (53) yields an equation of the form

$$\begin{aligned} Ra = Ra_0\big(1 &+ C_{11x}\delta_{Hx}^2 + C_{22x}\delta_{Vx}^2 + C_{33x}\varepsilon_{Hx}^2 + C_{44x}\varepsilon_{Vx}^2 \\ &+ C_{12x}\delta_{Hx}\delta_{Vx} + C_{34x}\varepsilon_{Hx}\varepsilon_{Vx} + C_{13x}\delta_{Hx}\varepsilon_{Hx} \\ &+ C_{24x}\delta_{Vx}\varepsilon_{Vx} + C_{14x}\delta_{Hx}\varepsilon_{Vx} + C_{23x}\delta_{Vx}\varepsilon_{Hx} \\ &+ C_{11y}\delta_{Hy}^2 + C_{22y}\delta_{Vy}^2 + C_{33y}\varepsilon_{Hy}^2 + C_{44y}\varepsilon_{Vy}^2 \\ &+ C_{12y}\delta_{Hy}\delta_{Vy} + C_{34y}\varepsilon_{Hy}\varepsilon_{Vy} + C_{13y}\delta_{Hy}\varepsilon_{Hy} \\ &+ C_{24y}\delta_{Vy}\varepsilon_{Vy} + C_{14y}\delta_{Hy}\varepsilon_{Vy} + C_{23y}\delta_{Vy}\varepsilon_{Hy}\big) \end{aligned} \tag{54}$$

A feature to be noted is that the effects of variation in the horizontal direction are decoupled from those in the vertical direction at this order of approximation. (For example, in Eq. (54) there is no term in the product $\delta_{Hx}\delta_{Hy}$.)

The numerical values of Ra_0 and the various coefficients, for some representative cases, are given in Tables 1 and 2. The first table is for the case of a square enclosure ($A = 1$), while the second table is for the case of what we call the optimal enclosure, where A is chosen to minimize the value of Ra_0. For the homogeneous case, Eq. (44) applies. This case corresponds to convection in the form of two-dimensional rolls in a horizontal layer of unlimited lateral extend. We comment on each of these tables in turn.

Table 2
Values of the Rayleigh number coefficients, defined by Eq. (54), for various parameter values (ξ, η) with $A = (\xi\eta)^{-0.25}$

ξ, η	0.1, 0.1	1, 0.1	10, 0.1	0.1, 1	1, 1	10, 1	0.1, 10	1, 10	10, 10
Ra_0	39.48	17.10	11.94	170.98	39.48	17.10	1194.22	170.98	39.48
C_{11x}	−0.200	−0.044	−0.010	−0.830	−0.200	−0.044	−2.984	−0.830	−0.200
C_{22x}	−0.801	−1.743	−4.076	−0.332	−0.801	−1.743	−0.119	−0.332	−0.801
C_{33x}	−0.200	−0.083	−0.040	−0.436	−0.200	−0.083	−1.019	−0.436	−0.200
C_{44x}	−0.200	−0.830	−2.984	−0.044	−0.200	−0.830	−0.010	−0.044	−0.200
C_{12x}	−1.001	−0.689	−0.509	−1.313	−1.001	−0.689	−1.492	−1.313	−1.001
C_{34x}	−0.500	−0.657	−0.746	−0.344	−0.500	−0.657	−0.254	−0.344	−0.500
C_{13x}	−0.400	−0.126	−0.040	−1.266	−0.400	−0.126	−4.002	−1.266	−0.400
C_{24x}	−0.801	−2.532	−8.006	−0.253	−0.801	−2.532	−0.080	−0.253	−0.801
C_{14x}	−0.320	−0.320	−0.320	−0.320	−0.320	−0.320	−0.320	−0.320	−0.320
C_{23x}	−0.640	−0.640	−0.640	−0.640	−0.640	−0.640	−0.640	−0.640	−0.640
C_{11y}	−0.343	−0.117	−0.024	−0.617	−0.343	−0.117	−0.810	−0.617	−0.343
C_{22y}	−0.086	−0.294	−0.593	−0.015	−0.086	−0.294	−0.002	−0.015	−0.086
C_{33y}	−0.022	−0.004	−0.001	−0.074	−0.022	−0.004	−0.148	−0.074	−0.022
C_{44y}	−0.343	−0.617	−0.810	−0.117	−0.343	−0.617	−0.024	−0.117	−0.343
C_{12y}	−0.429	−0.464	−0.296	−0.244	−0.429	−0.464	−0.101	−0.244	−0.429
C_{34y}	−0.214	−0.122	−0.050	−0.232	−0.214	−0.122	−0.148	−0.232	−0.214
C_{13y}	−0.086	−0.022	−0.004	−0.224	−0.086	−0.022	−0.398	−0.224	−0.086
C_{24y}	−0.172	−0.448	−0.795	−0.045	−0.172	−0.448	−0.008	−0.045	−0.172
C_{14y}	−0.274	−0.227	−0.127	−0.227	−0.274	−0.227	−0.127	−0.227	−0.274
C_{23y}	−0.034	−0.028	−0.016	−0.028	−0.034	−0.057	−0.016	−0.028	−0.034

The central column of Table 1 (for the square enclosure) is for the isotropic case, $\xi = \eta = 1$. The results agree with those published in [5]. There are some symmetries for this column and the two outside columns for which $\xi = \eta$: for example, $C_{11x} = C_{44x}$; this relates the variation of horizontal permeability to that of vertical conductivity. However, we do not place physical significance on this. For all the cases presented, the coefficients for the y-variation are all negative. That indicates that the vertical heterogeneity has a destabilizing effect. However, the horizontal heterogeneity has a stabilizing or destabilizing effect depending on the values of ξ and η. The positive coefficients correspond to cases when at least one of ξ and η is less than unity.

In the case of Table 2, for the optimal enclosure, the symmetrical patterns are more obvious. Indeed, the table shows a triplet and two doublets of identical columns. These illustrate the general result that for this enclosure the critical Rayleigh number is a function of the ratio η/ξ rather than of ξ, η independently. In the following way we have checked that this is indeed a general result.

Since the software package Mathematica can handle the algebraic expansion of a determinant of order 8, we have left it to do most of the algebraic work for us. However, it is worth noting that by means of elementary row and column transformations it is possible to manipulate the system Eqs. (35), (49)–(52) so as to put the determinant into quasi-diagonal form (so that terms off the principal diagonal are all small) and then directly expand the determinant up to terms of second order in the small quantities. In this way we were able to show that S takes the form

$$
\begin{aligned}
S = {} & (c_1\Delta_{Hx} + c_2\Delta_{Vx} + c_3\mathrm{E}_{Hx} + c_4\mathrm{E}_{Vx}) \\
& \times (c_5\Delta_{Hx} + c_6\Delta_{Vx} + c_7\mathrm{E}_{Hx} + c_8\mathrm{E}_{Vx}) \\
& + (c_9\Delta_{Hx} + c_{10}\Delta_{Vx})(c_{11}\mathrm{E}_{Hx} + c_{12}\mathrm{E}_{Vx}) \\
& + (d_1\Delta_{Hy} + d_2\Delta_{Vy} + d_3\mathrm{E}_{Hy} + d_4\mathrm{E}_{Vy}) \\
& \times (d_5\Delta_{Hy} + d_6\Delta_{Vy} + d_7\mathrm{E}_{Hy} + d_8\mathrm{E}_{Vy}) \\
& + (d_9\Delta_{Hy} + d_{10}\Delta_{Vy})(d_{11}\mathrm{E}_{Hy} + d_{12}\mathrm{E}_{Vy})
\end{aligned} \tag{55}
$$

From there it is straightforward to show that, for example, for the optimal enclosure

$$
C_{11x} = -\left(\frac{64}{9\pi^2}\right)\left[\frac{4\rho^2(4+\rho)}{3(1+\rho)(1+5\rho+\rho^2)}\right] \quad \text{where } \rho = \left(\frac{\eta}{\xi}\right)^{1/2} \tag{56}
$$

Similar expressions can be obtained for the other coefficients as functions of η/ξ. It follows that Ra is a function of η/ξ in the case of the optimal enclosure.

In fact, a referee has provided us with the following outline proof of this statement. If each element of $\mathbf{M}_{11}$, $\mathbf{M}_{12}$ is multiplied by ξ (something that leaves the determinant equation invariant) then ξ appears only as a multiplier of A, that is in the group ξA^2. Also, A appears in the form $\eta A^2 = (\eta/\xi)\xi A^2$ in the elements of $\mathbf{M}_{22}$. Substituting the value of A for the homogeneous case, then $\xi A^2 = (\xi/\eta)^{1/2}$. This leads to the determinant equation being a function of η/ξ, as required.

4. Conclusions

We have investigated a new extension of the Horton–Rogers–Lapwood problem, in which the horizontal and vertical heterogeneity of anisotropy of both permeability and thermal conductivity are considered. Our perturbation approach is pertinent to weak heterogeneity, in which each property varies by only a small fraction of its mean value. Within that restriction, our linear stability analysis is applicable to the case of a two-dimensional rectangular enclosure of arbitrary height-to-width aspect ratio. However, we have provided explicit results for just two situations, namely a square enclosure and an enclosure whose aspect ratio has been optimized to give the minimum Rayleigh number. In the latter case the permeability and conductivity ratios ξ and η affect the critical Rayleigh number via the ratio η/ξ. For the optimized enclosure all of the coefficients in the quadratic expression giving the increase in Rayleigh number are negative. In other words the effect of heterogeneity is always to reduce the critical Rayleigh number, in other words to reduce the stability. For the case of the square enclosure the same is generally true, but in some cases the horizontal heterogeneity can lead to an increase in stability.

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